SECTION 2.5: LIMITS AT INFINITY AND END BEHAVIOR

In Section 2.4, we studied infinite limits which corresponded graphically to vertical asymptotes. These features occurred when **outputs** from a function became **unbounded** near a given finite input.

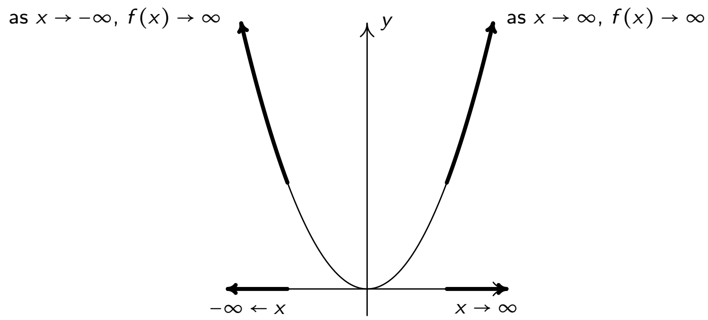
In this section, we study limits of the form $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$: limits where the **inputs** become unbounded.

You may recall from College Algebra that determining the behavior of a function $f(x)$ as $x \to \pm\infty$, that is, 'at the ends of the x-axis' is the concept of **end behavior**. As a refresher, consider the two cases below.

Looking at the table below on the left, we see that as the inputs, x, grow unbounded in the negative direction, the outputs, $f(x) = x^2$, grow unbounded in the positive direction. Graphically, the farther to the left we travel on the x-axis, the farther up the y-axis the function values travel. We describe this by writing: $\lim\limits_{x\to-\infty} x^2 = \infty$.

Similarly, we write $\lim\limits_{x\to\infty} x^2 = \infty$. since as the x values increase, so do the $f(x)$ values - seemingly without bound. This type of behavior occurs for all functions of the form $f(x) = x^n$ where $n = 2, 4, 6, \ldots$.

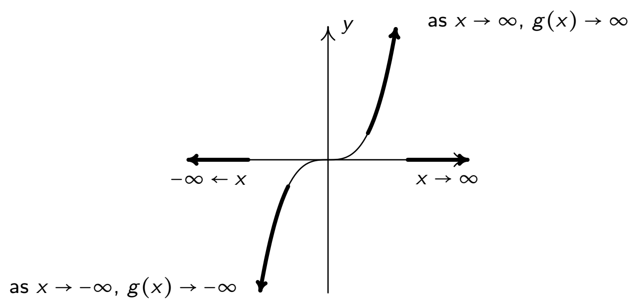
x	$f(x) = x^2$
-1000	1000000
-100	10000
-10	100
0	0
10	100
100	10000
1000	1000000



$f(x) = x^2$

Looking at $g(x) = x^3$, we find $\lim\limits_{x\to-\infty} x^3 = -\infty$ and $\lim\limits_{x\to\infty} x^3 = \infty$ These limits hold for all functions of the form $g(x) = x^n$ where $n = 1, 3, 5, \ldots$.

x	$g(x) = x^3$
-1000	-1000000000
-100	-1000000
-10	-1000
0	0
10	1000
100	1000000
1000	1000000000



$g(x) = x^3$

INFINITE LIMITS AT INFINITY: $\lim\limits_{x\to\infty} f(x) = \infty$ means... as the x values get large without bound, the function values $f(x)$ get large without bound.

QUESTION: How would you modify the verbiage above to define $\lim\limits_{x\to\infty} f(x) = -\infty$? $\lim\limits_{x\to-\infty} f(x) = \infty$? $\lim\limits_{x\to-\infty} f(x) = -\infty$?

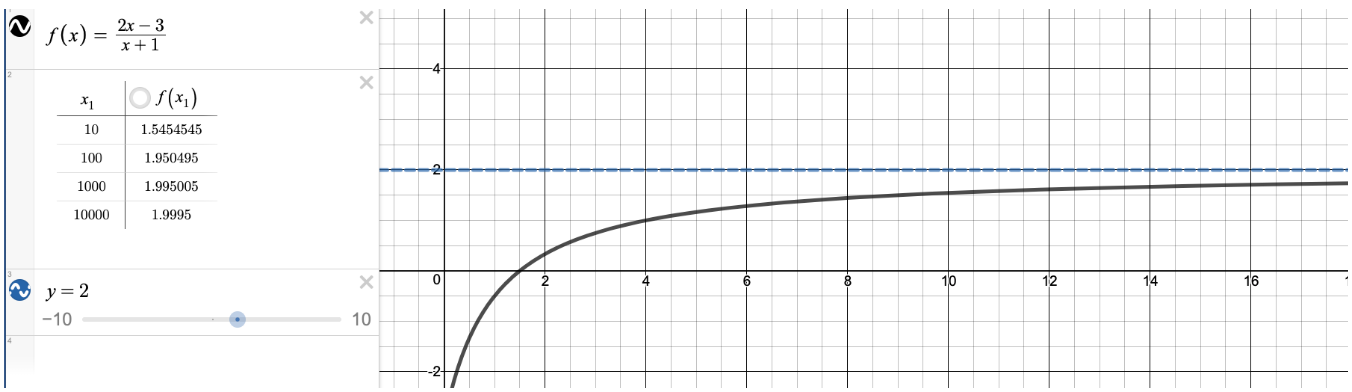
THEOREM: Powers of x at infinity:

- For any positive number p, $\lim_{x\to\infty} x^p = \infty$
- If n is a natural number, then:
  - If n is even, $\lim_{x\to-\infty} x^n = \infty$
  - If n is odd, $\lim_{x\to-\infty} x^n = -\infty$

We can understand the results of the theorem using 'number sense.' However, a more formal proof requires a more technical definition of limit (which we'll see later.)

EXAMPLE 1: Use tables and graphs to help you investigate $\lim_{x\to\infty} \frac{2x-3}{x+1}$.

Using desmos, we make a table and a graph and it certainly appears that $\lim_{x\to\infty} \frac{2x-3}{x+1} = 2$.



In this case, we say the line $y = 2$ is a **HORIZONTAL ASYMPTOTE** to the graph of $y = f(x)$.

More generally:

DEFINITION: $\lim_{x\to\infty} f(x) = L$ means... as the x values get large without bound, the function values $f(x)$ get closer and closer to L.

QUESTION: How would you modify the verbiage above to define $\lim_{x\to-\infty} f(x) = L$?

HORIZONTAL ASYMPTOTE: If one of $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, the line $y = L$ is called a **horizontal asymptote** to the graph of $y = f(x)$.

The following theorem is once again understood best using 'number sense.' We'll discuss this more formally later.

THEOREM: When defined, $\lim_{x\to\pm\infty} \frac{1}{x^p} = 0$ for any positive number, p.

Geometrically, this says the line $y = 0$ (a.k.a. the x-axis) is a horizontal asymptote of the graphs of $y = \frac{1}{x^p}$.

The following results will help us work with infinite limits and limits at infinity.

INFINITE ARITHMETIC:

PART ONE: Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = \infty$. Then:

- $\lim_{x\to c}[f(x) - g(x)] = -\infty$. That is, 'finite number $- \infty = -\infty$.'
- $\lim_{x\to c}[g(x) - f(x)] = \infty$. That is '$\infty -$ finite number $= \infty$.'
- If $L > 0$, then $\lim_{x\to c}[f(x)g(x)] = \infty$. That is, 'positive number $\cdot \infty = \infty$.'

  If $L < 0$, then $\lim_{x\to c}[f(x)g(x)] = -\infty$. That is, 'negative number $\cdot \infty = -\infty$.'

  If $L = 0$ and $f(x)$ is not identically 0 near $x = c$, then $\lim_{x\to c}[f(x)g(x)]$ gives the **indeterminate form** '$0 \cdot \infty$.'
- $\lim_{x\to c} \frac{f(x)}{g(x)} = 0$. That is, '$\frac{\text{finite number}}{\infty} = 0$.'

PART TWO: Suppose $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$. Then:

- $\lim_{x\to c}[f(x) + g(x)] = \infty$. That is, '$\infty + \infty = \infty$'.
- and $\lim_{x\to c}[f(x)g(x)] = \infty$. That is, '$\infty \cdot \infty = \infty$.'
- $\lim_{x\to c}[f(x) - g(x)]$ results in the **indeterminate form** '$\infty - \infty$,' requiring further analysis.
- $\lim_{x\to c} \frac{f(x)}{g(x)}$ results in the **indeterminate form** '$\frac{\infty}{\infty}$,' requiring further analysis.

EXAMPLE 2: Find the following limits.

- $\lim\limits_{x\to\infty}\left(x^2+3x-6\right)$

  Note, as $x\to\infty$, $x^2\to\infty$ and $3x\to\infty$.

  Hence, as $\lim\limits_{x\to\infty}\left(x^2+3x-6\right)$ is of the form '$\infty+\infty-$ finite number $=\infty$.'

  In other words, $\lim\limits_{x\to\infty}\left(x^2+3x-6\right)=\infty$.

- $\lim\limits_{x\to\infty}\left(x^2-3x-6\right)$

  As above, as $x\to\infty$, $x^2\to\infty$ and $3x\to\infty$.

  In this case, $\lim\limits_{x\to\infty}\left(x^2-3x-6\right)$ is of the form '$\infty-\infty-$ finite number' which is an indeterminate form.

  To help us see the forest for the trees, we factor out the highest power of x:

$$\lim_{x\to\infty}\left(x^2-3x-6\right)=\lim_{x\to\infty}x^2\left(1-\frac{3}{x}-\frac{6}{x^2}\right)$$

  As $x\to\infty$, $x^2\to\infty$ and $\left(1-\frac{3}{x}-\frac{6}{x^2}\right)\to 1+0+0=1$.

  Hence $\lim\limits_{x\to\infty}x^2\left(1-\frac{3}{x}-\frac{6}{x^2}\right)$ is of the form $\infty\cdot(\text{positive number})=\infty$, so $\lim\limits_{x\to\infty}\left(x^2-3x-6\right)=\infty$.

- $\lim\limits_{x\to-\infty}\left(4-x^2-2x^3\right)$.

  As $x\to-\infty$, $x^2\to\infty$ and $2x^3\to-\infty$. Hence, $\lim\limits_{x\to-\infty}\left(4-x^2-2x^3\right)$ is of the form 'finite number $-\infty+\infty$,' which is an indeterminate form. Taking a cue from the previous example, we factor:

$$\lim_{x\to-\infty}\left(4-x^2-2x^3\right)=\lim_{x\to-\infty}x^3\left(\frac{4}{x^3}-\frac{1}{x}-2\right)$$

  As $x\to-\infty$, $x^3\to-\infty$ whereas $\left(\frac{4}{x^3}-\frac{1}{x}-2\right)\to 0-0-2=-2$.

  Hence, $\lim\limits_{x\to-\infty}x^3\left(\frac{4}{x^3}-\frac{1}{x}-2\right)$ is of the form $(-\infty)\cdot(\text{negative number})=\infty$, so $\lim\limits_{x\to-\infty}\left(4-x^2-2x^3\right)=\infty$.

Using the idea from the last two last examples, we can prove a fact that you may recall from College Algebra:

LEADING TERM TEST FOR POLYNOMIAL FUNCTIONS:

The end behavior of polynomial functions is determined by the leading term. More formally:

If $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$ is a polynomial function of degree n then:

$$\lim_{x\to\pm\infty}f(x)=\lim_{x\to\pm\infty}a_nx^n$$

The idea of the proof is to factor: $a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0=x^n\left(a_n+\frac{a_{n-1}}{x}+\ldots+\frac{a_2}{x^{n-2}}+\frac{a_1}{x^{n-1}}+\frac{a_0}{x^n}\right)$.

Terms with an 'x' in the denominator go to 0 as $x\to\pm\infty$ leaving a_nx^n as the only term which affects the limit.

Our next example revisits $\lim\limits_{x\to\infty} \dfrac{2x-3}{x+1}$ which we used tables and graphs to 'guesstiamate' is 2.

EXAMPLE 3: Use limit properties to determine $\lim\limits_{x\to\infty} \dfrac{2x-3}{x+1}$.

We can use the leading term test for polynomials to perform a quick analysis of the situation.

The leading term in the numerator, $2x-3$ is '$2x$' while the leading term in the denominator, $x+1$ is 'x'.

Hence, as $x \to \infty$, $\dfrac{2x-3}{x+1} \approx \dfrac{2x}{x} = 2$.

To arrive at this answer more formally, we employ the strategy in the previous example and factor out the highest power of x from both the numerator and denominator:

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x-3}{x+1} &= \lim_{x\to\infty} \frac{x\left(2-\frac{3}{x}\right)}{x\left(1+\frac{1}{x}\right)} \\
&= \lim_{x\to\infty} \frac{\cancel{x}\left(2-\frac{3}{x}\right)}{\cancel{x}\left(1+\frac{1}{x}\right)} \\
&= \lim_{x\to\infty} \frac{2-\frac{3}{x}}{1+\frac{1}{x}} \\
&= \frac{\lim\limits_{x\to\infty}\left(2-\dfrac{3}{x}\right)}{\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)} \\
&= \frac{2-3(0)}{1+0} \\
\lim_{x\to\infty} \frac{2x-3}{x+1} &= 2
\end{aligned}$$

Which confirms what we suspected using tables and graphs and what we determined using the leading term test.

EXAMPLE 4: Find the following limit and interpret graphically: $\lim\limits_{x\to\infty} \dfrac{4-x}{x^2+1}$

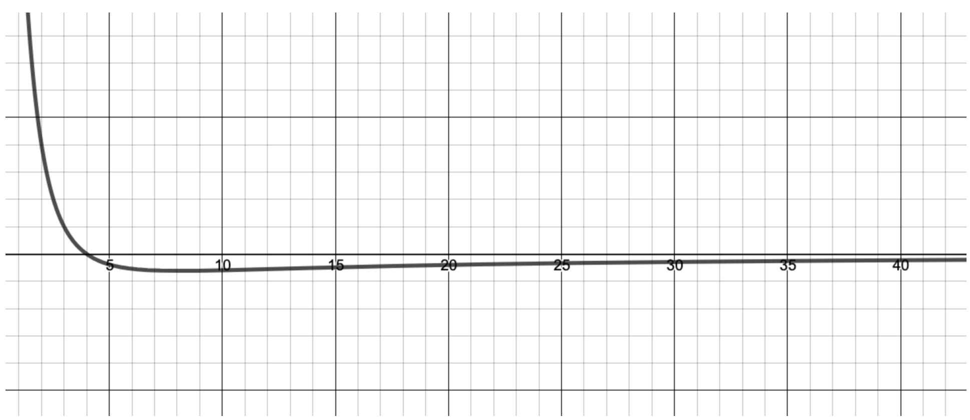
Using the leading term test for a quick analysis, we have that as $x \to \infty$, $\dfrac{4-x}{x^2+1} \approx \dfrac{-x}{x^2} = -\dfrac{1}{x} \to 0$.

More formally,

$$\begin{aligned}
\lim_{x\to\infty} \frac{4-x}{x^2+1} &= \lim_{x\to\infty} \frac{x\left(\frac{4}{x}-1\right)}{x^2\left(1+\frac{1}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{\cancel{x}\left(\frac{4}{x}-1\right)}{x^{\cancel{2}\,1}\left(1+\frac{1}{x^2}\right)} \\
&= \lim_{x\to\infty} \left(\frac{1}{x}\right)\frac{\left(\frac{4}{x}-1\right)}{\left(1+\frac{1}{x^2}\right)} \\
&= 0\cdot\frac{(0-1)}{(1+0)} \\
\lim_{x\to\infty} \frac{4-x}{x^2+1} &= 0.
\end{aligned}$$

Graphically, $y=0$ is a horizontal asymptote to the graph of $y=\dfrac{4-x}{x^2+1}$ as $x\to\infty$.

NOTE: The graph of $y=f(x)$ actually crosses the horizontal asymptote at $x=4$.



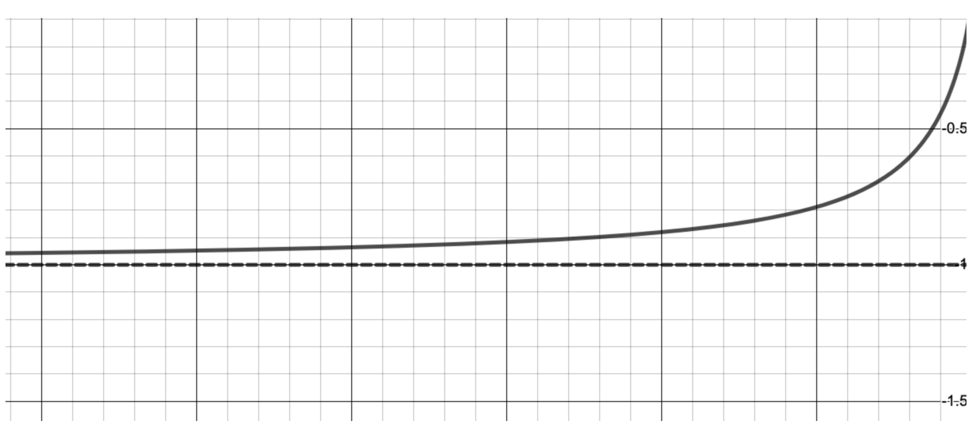
EXAMPLE 5: Find the following limit and interpret graphically: $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{4x^2-x}}{2x-3}$

Once again, the leading term test gives as $x\to-\infty$, $\dfrac{\sqrt{4x^2-x}}{2x-3}\approx\dfrac{\sqrt{4x^2}}{2x}=\dfrac{2|x|}{2x}=\dfrac{|x|}{x}$.

Since $x\to-\infty$, $x<0$ so $|x|=-x$ and $\dfrac{|x|}{x}=\dfrac{-x}{x}=-1$. Hence we expect $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{4x^2-x}}{2x-3}=-1$.

$$
\begin{aligned}
\lim_{x\to-\infty}\frac{\sqrt{4x^2-x}}{2x-3} &= \lim_{x\to-\infty}\frac{\sqrt{x^2\left(4-\frac{x}{x^2}\right)}}{x\left(2-\frac{3}{x}\right)}\\
&= \lim_{x\to-\infty}\frac{|x|\sqrt{4-\frac{1}{x}}}{x\left(2-\frac{3}{x}\right)}\\
&= \lim_{x\to-\infty}\frac{(-x)\sqrt{4-\frac{1}{x}}}{x\left(2-\frac{3}{x}\right)} \qquad \text{Since } x\to--\infty,\ x<0 \text{ so } |x|=-x.\\
&= \lim_{x\to-\infty}\frac{(-\cancel{x})\sqrt{4-\frac{1}{x}}}{\cancel{x}\left(2-\frac{3}{x}\right)}\\
&= \lim_{x\to-\infty}\frac{-\sqrt{4-\frac{1}{x}}}{\left(2-\frac{3}{x}\right)}\\
&= \frac{-\sqrt{4-0}}{2-0}\\
\lim_{x\to-\infty}\frac{\sqrt{4x^2-x}}{2x-3} &= -1
\end{aligned}
$$

Graphically, $y=-1$ is a horizontal asymptote to the graph of $y=\dfrac{4-x}{x^2+1}$ as $x\to-\infty$.



QUESTION: How would you adjust the work above to determine $\displaystyle\lim_{x\to\infty}\frac{\sqrt{4x^2-x}}{2x-3}$? What is $\displaystyle\lim_{x\to\infty}\frac{\sqrt{4x^2-x}}{2x-3}$?

Check your answer using desmos.

In addition to the arithmetic limit properties holding for limits at infinity, the Squeeze Theorem also carries over.

EXAMPLE 6: Use the Squeeze Theorem to determine: $\lim\limits_{t\to\infty} \dfrac{2t+\sin(3t)}{t}$

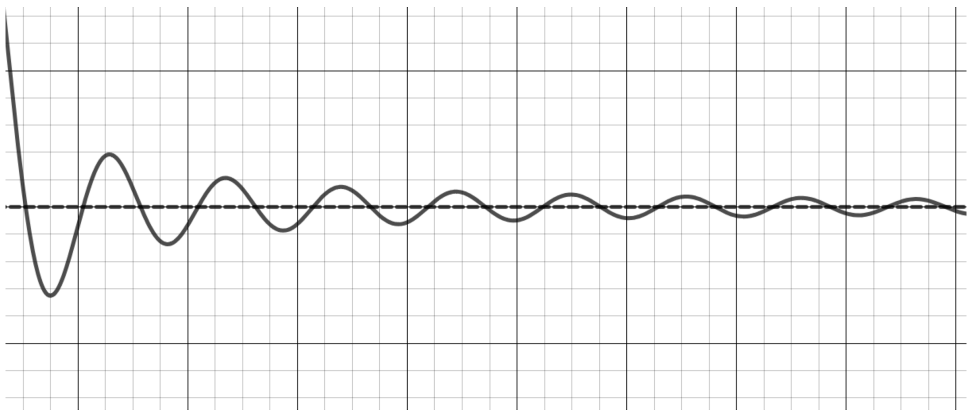
We first note that $\lim\limits_{t\to\infty} \sin(3t)$ does not exist since $\sin(3t)$ continues to oscillate forever more.

That being said, we know that $-1 \le \sin(3t) \le 1$, which means $\dfrac{2t-1}{t} \le \dfrac{2t+\sin(3t)}{t} \le \dfrac{2t+1}{t}$.

We find $\lim\limits_{t\to\infty} \dfrac{2t-1}{t} = \lim\limits_{t\to\infty} \left(2-\dfrac{1}{t}\right) = 2-0 = 2$ and $\lim\limits_{t\to\infty} \dfrac{2t+1}{t} = \lim\limits_{t\to\infty} \left(2+\dfrac{1}{t}\right) = 2+0 = 2$.

Hence, by the Squeeze Theorem, $\lim\limits_{t\to\infty} \dfrac{2t+\sin(3t)}{t} = 2$.

Geometrically, $y = 2$ is a horizontal asymptote to $y = \dfrac{2t+\sin(3t)}{t}$ as $t \to \infty$.



Looking a bit more precisely, we may write $\dfrac{2t+\sin(3t)}{t} = 2 + \frac{1}{t}\sin(3t) = \frac{1}{t}\sin(3t) + 2$.

This has the form of a sinusoid with vertical shift $y = 2$ and variable amplitude $\frac{1}{t}$ which goes to 0 as $t \to \infty$.

EXAMPLE 7: Determine $\lim\limits_{x\to\infty} \dfrac{4-x-x^3}{x^2-2x+3}$.

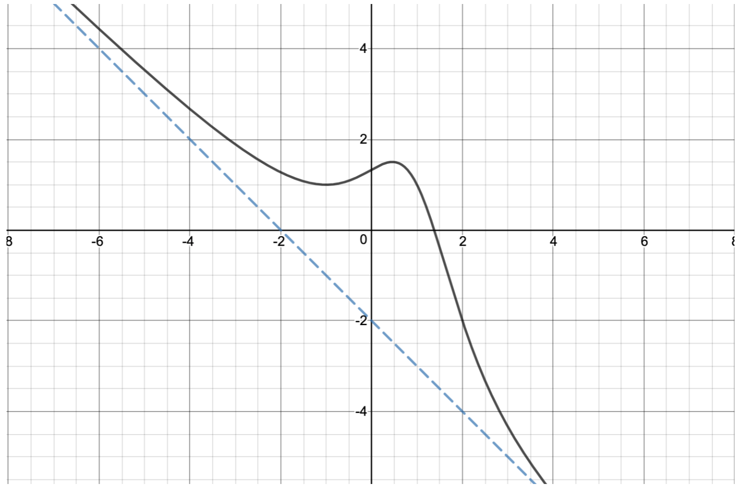
As usual, we make a quick analysis using the leading term test: as $x \to \infty$, $\dfrac{4-x-x^3}{x^2-2x+3} \approx \dfrac{-x^3}{x^2} = -x \to -\infty$.

Formally,

$$\begin{aligned}
\lim_{x\to\infty} \frac{4-x-x^3}{x^2-2x+3} &= \lim_{x\to\infty} \frac{x^3\left(\frac{4}{x^3}-\frac{1}{x^2}-1\right)}{x^2\left(1-\frac{2}{x}+\frac{3}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{x^{\cancel{3}\,1}\left(\frac{4}{x^3}-\frac{1}{x^2}-1\right)}{\cancel{x^2}\left(1-\frac{2}{x}+\frac{3}{x^2}\right)} \\
&= \lim_{x\to\infty} x\,\frac{\left(\frac{4}{x^3}-\frac{1}{x^2}-1\right)}{\left(1-\frac{2}{x}+\frac{3}{x^2}\right)}
\end{aligned}$$

As $x \to \infty$, $\dfrac{\left(\frac{4}{x^3}-\frac{1}{x^2}-1\right)}{\left(1-\frac{2}{x}+\frac{3}{x^2}\right)} \to \dfrac{0-0-1}{1-0+0} = -1$, hence, $\lim\limits_{x\to\infty} x\,\dfrac{\left(\frac{4}{x^3}-\frac{1}{x^2}-1\right)}{\left(1-\frac{2}{x}+\frac{3}{x^2}\right)}$ is of the form $\infty\cdot$(negative number) $= -\infty$.

Graphing $f(x) = \dfrac{4 - x - x^3}{x^2 - 2x + 3}$ to check our answer, we see that as $x \to \pm\infty$, $y = f(x)$ resembles a line.



DEFINITION: A line $y = mx + b$ where $m \neq 0$ is called a **slant** or an **oblique** asymptote to the graph of $y = f(x)$ if one or both of the following are true:

$$\lim_{x\to\infty} [f(x) - (mx + b)] = 0 \qquad \text{or} \qquad \lim_{x\to-\infty} [f(x) - (mx + b)] = 0$$

In our scenario, it certainly **appears** as if the graph of $y = f(x)$ has a slant asymptote - but how do we **prove** it?

From our leading term analysis, we found as $x \to \pm\infty$, $f(x) = \dfrac{4 - x - x^3}{x^2 - 2x + 3} \approx -x$.

It stands to reason that if we have a slant asymptote, $y = mx + b$, then $m = -1$. How do we determine b?

Let's consider the limit: $\lim\limits_{x\to\infty} [f(x) - (-x)]$.

$$\begin{aligned}
\lim_{x\to\infty} [f(x) - (-x)] &= \lim_{x\to\infty} \left[\frac{4 - x - x^3}{x^2 - 2x + 3} - (-x)\right] \\
&= \lim_{x\to\infty} \left[\frac{4 - x - x^3}{x^2 - 2x + 3} + x\right] \\
&= \lim_{x\to\infty} \left[\frac{4 - x - x^3}{x^2 - 2x + 3} + \frac{x\left(x^2 - 2x + 3\right)}{x^2 - 2x + 3}\right] \\
&= \lim_{x\to\infty} \left[\frac{4 - x - x^3 + x^3 - 2x^2 + 3x}{x^2 - 2x + 3}\right] \\
&= \lim_{x\to\infty} \left[\frac{-2x^2 + 2x + 4}{x^2 - 2x + 3}\right] \\
&= -2
\end{aligned}$$

Since $\lim\limits_{x\to\infty} [f(x) - (-x)] = -2$, it follows[1] that $\lim\limits_{x\to\infty} [f(x) - (-x - 2)] = 0$.

This is precisely what we mean by saying $y = -x - 2$ is the slant asymptote to the graph of $y = f(x)$!

[1] Do you see why?

STRATEGY FOR FINDING SLANT ASYMPTOTES:

Suppose the Leading Term Test gives $f(x) \approx mx$ as $x \to \infty$ or $x \to -\infty$, and $b = \lim_{x\to\infty}[f(x) - mx]$.

Then $y = mx + b$ is a slant asymptote to the graph of $y = f(x)$.

EXAMPLE 8: Find the slant asymptote(s) to the graph of $f(x) = \sqrt{4x^2 + x + 1}$.

Using the leading term test, we know as $x \to \pm\infty$, $f(x) = \sqrt{4x^2 + x + 1} \approx \sqrt{4x^2} = 2|x|$.

If $x \to \infty$, then $|x| = x$ so $f(x) \approx 2x$. Hence, we proceed to find: $\lim_{x\to\infty}[f(x) - 2x] = \lim_{x\to\infty}\left[\sqrt{4x^2 + x + 1} - 2x\right]$:

Note that we have the indeterminate form '$\infty - \infty$' here so we need to do some algebra to help resolve this.

Conjugates to the rescue!

$$
\begin{aligned}
\lim_{x\to\infty}\left[\sqrt{4x^2+x+1} - 2x\right] &= \lim_{x\to\infty}\left(\sqrt{4x^2+x+1} - 2x\right) \cdot \frac{\sqrt{4x^2+x+1} + 2x}{\sqrt{4x^2+x+1} + 2x} \\
&= \lim_{x\to\infty} \frac{\left(4x^2+x+1\right) - 4x^2}{\sqrt{4x^2+x+1} + 2x} \\
&= \lim_{x\to\infty} \frac{x+1}{\sqrt{4x^2+x+1} + 2x} \\
&= \lim_{x\to\infty} \frac{x+1}{\sqrt{x^2\left(4 + \frac{1}{x} + \frac{1}{x^2}\right)} + 2x} \\
&= \lim_{x\to\infty} \frac{x+1}{|x|\sqrt{4 + \frac{1}{x} + \frac{1}{x^2}} + 2x} \\
&= \lim_{x\to\infty} \frac{x+1}{x\sqrt{4 + \frac{1}{x} + \frac{1}{x^2}} + 2x} && |x| = x \text{ since } x \to \infty \text{ so } x > 0 \\
&= \lim_{x\to\infty} \frac{x\left(1 + \frac{1}{x}\right)}{x\left(\sqrt{4 + \frac{1}{x} + \frac{1}{x^2}} + 2\right)} \\
&= \lim_{x\to\infty} \frac{\cancel{x}\left(1 + \frac{1}{x}\right)}{\cancel{x}\left(\sqrt{4 + \frac{1}{x} + \frac{1}{x^2}} + 2\right)} \\
&= \frac{1}{\sqrt{4} + 2} = \frac{1}{4}
\end{aligned}
$$

Hence, as $x \to \infty$, $y = 2x + \frac{1}{4}$ is a slant asymptote, which we invite you to verify on desmos.

NOTE: As the above calculation shows, we must exercise caution when applying the leading term test here!

As $x \to \infty$, $\sqrt{4x^2 + x + 1} \approx \sqrt{4x^2} = |2x| = 2x$ so you may be lead to believe that $\sqrt{4x^2 + x + 1} - 2x \approx 2x - 2x \to 0$.

This is incorrect, however since we have the indeterminate form: $\infty - \infty$.

To determine the slant asymptote as $x \to -\infty$, we can re-use much of the same work above!

The key difference being that if $x \to -\infty$, then $|x| = -x$ so $f(x) \approx -2x$.

We leave it to you to show $\lim\limits_{x \to -\infty} [f(x) - (-2x)] = \lim\limits_{x \to -\infty} \left[\sqrt{4x^2 + x + 1} + 2x\right] = -\dfrac{1}{4}$.

Hence, as $x \to -\infty$, $y = -2x - \dfrac{1}{4}$ is a slant asymptote.

EXAMPLE 9 (VIDEO): Use limits to find the horizontal and slant asymptotes of the graphs of the following.

1. $f(x) = \dfrac{2x^2 - 3x + 1}{1 - x^2}$ — Ans: HA: $y = -2$

2. $f(x) = \dfrac{1 - 3x}{(x - 2)^2}$ — Ans: HA: $y = 0$

3. $f(x) = \dfrac{6x}{\sqrt{9x^2 - 2x + 1}}$ — HAs: $y = -2$ and $y = 2$

4. $f(x) = \dfrac{2x^2 - 3x + 1}{1 - x}$ — SA: $y = -2x + 1$

5. $f(x) = \dfrac{(x - 2)^2}{1 - 3x}$ — SA: $y = -\dfrac{1}{3}x - \dfrac{11}{9}$

6. $f(x) = \dfrac{6x^2}{\sqrt{9x^2 - 2x + 1}}$ — SAs: $y = -2x - \dfrac{2}{9}$ and $y = 2x + \dfrac{2}{9}$

END BEHAVIOR OF RATIONAL FUNCTIONS FROM PRECALCULUS (REVISITED):

You probably learned the rules below as they pertain specifically to **rational** functions in precalculus. It's worth taking some to time to revisit this result using the (more general) Calculus tools we've developed in this section.

- **HORIZONTAL ASYMPTOTES** occur in one of two cases:
  - If the degree of the numerator is **less** than the degree of the denominator, the HA is $y = 0$.
  - If the degree of the numerator is the **same** as the degree of the denominator, the HA is:

$$y = \frac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}$$

- **SLANT (OBLIQUE) ASYMPTOTES** occur if the degree of the numerator is **exactly one more** than the degree of the denominator. To find the asymptote, perform **long division** and discard the remainder.

HOMEWORK: Section 2.5: 1 - 73 every other odd, 77*, 81*